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THE SIMULATION OF UNSTEADY TRANSONIC FLOW AND THE STABILITY OF A TRANSONIC BOUNDARY LAYER†

A. N. BOGDANOV and V. N. DIYESPEROV

Moscow

e-mail: bogdanov@imec.msu.ru

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A modified "triple-deck" model is proposed to investigate free unsteady viscid–inviscid interaction at transonic velocities. The modification of the model is the inclusion of the singular term of the transonic expansion, thanks to which the Lin–Reissner–Tsien equation can be refined and a more accurate description given of unsteady and non-linear phenomena. The modified model is used to consider the stability of the boundary layer interacting with an unsteady inviscid flow at transonic velocities. It is shown that the modified model enables one to take into account an additional perturbation, which escapes consideration when the classical triple-deck model is used. © 2005 Elsevier Ltd. All rights reserved.

In the model describing free unsteady viscid–inviscid interaction of the boundary layer and the outer flow at transonic velocities [1], which is analogous to free viscid–inviscid interaction in steady supersonic flow [2], the flow field is divided into three parts, referred to as "decks" (the "triple-deck" model). Solutions of the equations describing the flow, derived separately for each deck using asymptotic expansions in the approximation of large Reynolds numbers, are matched using the boundary conditions at the deck boundaries.

It has been shown that for the principal terms of the expansions in the first deck, also known as the outer or upper deck, the flow is inviscid and unsteady and is described by the Lin–Reissner–Tsien (LRT) equation for the velocity potential. In the second (main or middle) deck, which occupies the main part of the boundary layer, the flow in the first approximation is inviscid, vortical and steady. In the third (wall) deck, the flow is described in the first approximation by the steady equations of the boundary layer of an incompressible fluid. It has been proposed [1] to use these equations to analyse unsteady processes in the free viscid–inviscid interaction of the boundary layer with the outer flow in the transonic velocity range. The system of equations obtained has been used to find a similarity law for flows of this kind.

As to the results obtained, a few remarks are in order. The LRT equation is known [3] to have certain drawbacks, which make it impossible to describe correctly the propagation in a flow of unsteady perturbations: the LRT equation yields infinite velocities of propagation of weak perturbations downstream; the wave fronts of the perturbations from a point source are non-closed curves (parabolas) at all instants of time, the Cauchy problem for such an equation is ill-posed; the LRT equation does not describe high-frequency perturbations, and so on.

A modification of the equations of free unsteady viscid–inviscid interaction at transonic velocities will be proposed below. It involves retaining the term with the second time derivative when deriving the equation of transonic flow from the complete equations for the potential. Although this term is small to a higher order, it is singular in the equation obtained, and ignoring it leads to degeneration of the equation (one of the fundamental equations in the flow model) into a degenerate hyperbolic equation (like the LRT equation) and to the appearance of the aforementioned drawbacks of the model. Inclusion of this term yields a more accurate flow model and a better description of unsteady and non-linear phenomena.

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The modified model will also be used below to analyse the behaviour of linear and weakly non-linear perturbations. The possibility of perturbation growth, which is overlooked in the traditional analysis, will be demonstrated. Wave fronts of weak perturbations will be constructed. The formation of a shock wave from a weakly non-linear perturbation will be investigated.

The problem of the stability of the boundary layer in free viscid–inviscid interaction has been studied in detail, within the framework of the triple-deck model, by Ryzhov and his disciples [4–9]. At first it was assumed that the outer flow was steady; the cases of supersonic and subsonic velocities of the outer flow were considered as special problems.

For interaction with a supersonic outer flow, a dispersion relation (DR) was obtained and the results of a detailed computer analysis of the DR were given [4]. The DR was derived and analysed for the case of interaction with a subsonic flow [5];† the presence of a growing perturbation was observed for subsonic interaction regimes but not for supersonic ones.

It has been shown [6, 7] that there is an infinite number of perturbation waves that travel downstream in the boundary layer, but only one wave propagating upstream. It has also been observed [8] that the problem of the stability of the boundary layer in viscid–inviscid interaction reduces to the same DR as is obtained in the problem of the development of long-wave perturbations in a viscous incompressible fluid. Thus, perturbations in an interacting boundary layer are Tollmien–Schlichting waves, and they can be analysed using Prandtl's equation instead of the complete Navier–Stokes equations.

The stability of a boundary layer freely interacting with an unsteady transonic outer flow was analysed in [9]. In this connection, the systems of equations describing the flow have been investigated and it has been shown that, when deriving them, one has to choose between allowance for the unsteady nature of the flow in the lower deck (boundary layer) and allowance for the non-linearity of the flow in the upper deck (outer inviscid flow). (In non-Russian literature, such as [10], such models have been called transonic regimes of the first and second kinds.) In investigating stability for the velocity potential of the outer flow, the linear LRT equation has been used [9]. It has been found [9] that the DR has one increasing root, irrespective of whether the velocity of the outer flow exceeds the velocity of sound; the growth rate of the perturbations depends on the magnitude of the wave number. Comparison with the results of research into interactions at subsonic velocities has shown [9] that the results contradict the transition, familiar in subsonic interaction regimes, to stable damped oscillations as the wave number increases.

Other models are presently being proposed to investigate the processes that occur in free unsteady viscid–inviscid interaction. For the case in which the perturbations of the flow field exceed the magnitude permitted by the triple-deck model, a "quadruple-deck" model has been proposed [11]. In that model it has been possible to construct soliton solutions of the equations of free unsteady viscid–inviscid interaction at transonic velocities (in the interaction region at subsonic velocities).

Comparison of experimental and theoretical results of research on the propagation of perturbations in free viscid–inviscid interaction, based on the triple-deck model, indicates good qualitative agreement [12]. The triple-deck model correctly predicts all the qualitative features of the radiation of unstable oscillations: the propagation of perturbations upstream, the local maximum of the perturbation amplitude above a vibrator with less marked local minimum downstream, and the excitation of Tollmien–Schlichting waves. Possible reasons for the quantitative disagreement of the results (the numerical values of the characteristic frequencies may differ by a factor of 2–3) are as follows: the three-dimensional nature of unstable oscillations, while the theory is based on a two-dimensional approach, large variations of local Reynolds numbers as the perturbations propagate farther from the source, corresponding in the theory to transition from the downstream to upstream branch of the neutral stability curve, etc. No special differences between experiment and theory in the development of perturbations downstream were noted in [12].

The stability of a boundary layer freely interacting with an unsteady inviscid flow at transonic velocities will also be considered below. To describe the flow field, the triple-deck model will be used; a modified linear LRT equation will be used to simulate the outer unsteady transonic flow [3]. The use of this equation enables us to overcome the drawbacks of the simulation of unsteady transonic flow using the classical transonic approximation [13]. In particular, unlike previous research, it has been possible to make allowance for the propagation of unsteady perturbations in the outer flow both upstream and downstream. A DR has been obtained that includes the cases previously considered, and its roots are determined. It has been shown that the modified model enables one to allow for the downstream

† For an extension of this research see Ye. D. TERENT'YEV, Unsteady boundary-layer problems with free interaction. Doctorate dissertation, 1.2.05, Moscow, 1986.

propagation of a perturbation that amplifies in time, which escapes consideration when the traditional model is used [9].

1. A MODIFIED TRIPLE-DECK MODEL OF FREE UNSTEADY VISCID–INVISCID INTERACTION AT TRANSONIC VELOCITIES

We shall assume that the outer flow (far from the surface around which the flow is taking place) is inviscid and unsteady; for its parameters we shall use the expansions [1]

$$\begin{gathered} t = L(t_0 + \varepsilon^{4/5} t_1)/U_\infty, \quad x = L(1+\varepsilon^{12/5} x_1), \quad y = L\varepsilon^{8/5} y_1 \\ u = U_\infty(1+\varepsilon^{8/5} u_{11} + \varepsilon^{16/5} u_{12} + \ldots), \quad \upsilon = U_\infty(\varepsilon^{12/5}\upsilon_{11} + \varepsilon^{20/5}\upsilon_{12} + \ldots) \\ \rho = \rho_\infty(1+\varepsilon^{8/5}\rho_{11} + \varepsilon^{16/5}\rho_{12} + \ldots), \quad p = p_\infty + \rho_\infty U_\infty^2(\varepsilon^{8/5} p_{11} + \varepsilon^{8/5} p_{12} + \ldots) \end{gathered} \tag{1.1}$$

where p is the pressure, ρ is the density, u and υ are the components of the velocity along the axes of the space coordinates x and y (respectively), and t is the time. The terms of the expansions f_{11} are the functions corresponding to the first "deck" of coordinates (x_1, y_1, t_1), ε is a small parameter, which depends on the Reynolds number $\mathrm{Re} = \rho_\infty L U_\infty/\nu_\infty$, L is the characteristic length, U is the velocity of the unperturbed flow, ν is the coefficient of kinematic viscosity, and the subscript ∞ indicates characteristic values of the parameters (constant quantities). In the lower decks of the modified model, the appropriate equations of the conventional triple-deck model [1] will be used.

At transonic velocities, the difference $M^2 - 1$, where M is the Mach number, is a small quantity. When expansions (1.1) are used, $M^2 - 1$ is $O(\varepsilon^{8/5})$. Substitution of expansions (1.1) into the equations of a viscous compressible gas [14], taken here as the initial equations, yield the relations of acoustics for the principal terms ($O(\varepsilon^{-4/5})$) (it is assumed that the free stream is homogeneous and constant)

$$p_{11} = \rho_{11} = -u_{11} \tag{1.2}$$

The next approximation (for $O(\varepsilon^{4/5})$ terms) yields the Lin–Reissner–Tsien (LRT) equation for the flow velocity potential (see, e.g., [13]), which is familiar from gas dynamics and used by many authors when analysing unsteady transonic flow. This equation, however, possesses several serious drawbacks (see above). Note that the LRT equation is a characteristic limit: there are no other expansions that yield more general results for this problem. This situation determines (through matching conditions) the form of the expansion of the parameters as functions of time and the longitudinal coordinate in other regions of the flow (for the principal and boundary-layer decks).

One way out of the difficulties is to replace the LRT equation by a modified equation [3], which differs from the former in having a term with a second time derivative, which enables the analysis of unsteady and steady phenomena in the flow to be refined.

The modified LRT equation [3] for the velocity potential $\phi(u_{11} = \partial\phi/\partial x_1, \upsilon_{11} = \partial\phi/\partial y_1)$ has the following form ($\delta = \varepsilon^{8/5} = \mathrm{Re}^{-1/5}$; γ is the Poisson adiabatic exponent)

$$\delta\frac{\partial^2\phi}{\partial t_1^2} + 2\frac{\partial^2\phi}{\partial t_1 \partial x_1} + K^*\frac{\partial^2\phi}{\partial x_1^2} - \frac{\partial^2\phi}{\partial y_1^2} = 0, \quad K^* = (1-M_\infty^2)\delta^{-1} - (\gamma+1)\frac{\partial\phi}{\partial x_1} \tag{1.3}$$

Equation (1.3) and its applications to problems of unsteady transonic aerodynamics have been investigated in detail in [3, 15]. The simulation of unsteady transonic flow on the basis of the modified equation (1.3) yields a picture in better agreement with the physical nature of the flow. Unlike the LRT equation, Eq. (1.3) (and this is its main advantage) does not have degenerate characteristic directions and it describes (though to a different degree of accuracy) unsteady perturbations propagating in any direction in the flow field.

Previous analysis [3] of the behaviour of linear and weakly non-linear unsteady perturbations based on Eq. (1.3) have yielded the following results.

For linear harmonic perturbations (ω is the frequency and k is the wave number)

$$\phi \sim \exp(i(-\omega t + kx + ly))$$

of a steady flow with slowly varying parameters, the roots of the dispersion relation (DR) are determined

$$\omega_1 = 2\frac{k}{\delta} + \frac{k}{2}\bar{\Omega}, \quad \omega_2 = -\frac{k}{2}\bar{\Omega}; \quad \bar{\Omega} = K^{*0} + \frac{i}{k}(\gamma + 1)\frac{\partial^2 \phi^0}{\partial x_1^2} + \left(\frac{l}{k}\right)^2 \tag{1.4}$$

A superscript zero indicates the parameters of the unperturbed flow. As $\delta \to 0$, the root ω_1 disappears, and analysis of the corresponding downstream-propagating perturbations is impossible. At the same time, as is clear from (1.4), the imaginary part of the roots corresponding to growth (damping) of the perturbation depends on the nature of the unperturbed flow (on quantity $\partial^2\phi^0/\partial x_1^2$). The roots ω_1 and ω_2 have different signs, that is, it follows that the root that is dropped as $\delta \to 0$, if $\partial^2\phi^0/\partial x_1^2 > 0$, will correspond to a wave with increasing amplitude ($\mathrm{Im}\,\omega_1 > 0$), whereas the other root corresponds to a damped wave ($\mathrm{Im}\,\omega_2 < 0$).

The characteristic surfaces for a point source of perturbations at the point (x_0, y_0) are defined by the equation

$$K^*t^2 + 2t(x - x_0) - \delta(x - x_0)^2 = (y - y_0)^2(1 + \delta K^*) \tag{1.5}$$

which, if $K^* = \text{const}$, defines an ellipse in the plane $t = \text{const}$.

To simplify matters, let us put $x_0 = y_0 = 0$, ignore the quantity δK^*, and transform Eq. (1.5) to the form

$$y^2 + \delta(x - \delta^{-1}t)^2 = (K^* + \delta^{-1})t^2 \tag{1.6}$$

analogous to the equation of the characteristic front of the weak perturbation defined by the wave equation in a moving medium (a is the velocity of sound)

$$y^2 + (x - Ut)^2 = a^2t^2$$

It is obvious from Eq. (1.6) that in this model the velocity of the unperturbed flow carrying the weak perturbation downstream is δ^{-1}. The coefficient $K^* + \delta^{-1}$ characterizes the velocity at which the front of such a perturbation propagates. As $\delta \to 0$, the velocity at which the perturbation is carried away becomes infinite, as does the velocity which the perturbation front propagates downstream. The sum of the upstream velocity of the moving front of the perturbation and the velocity of the unperturbed flow carrying it gives a finite velocity, while the parts of the perturbation front propagating downstream tend to infinity, and the ellipse (1.5) degenerates to a parabola in the plane $t = \text{const}$ [3, 13].

Reversal of the part of the characteristic front generated by the perturbation implies the formation of a shock wave. The condition for reversal (like the condition for the intersection of characteristic fronts, which corresponds to the onset of reversal) may be determined by Eq. (1.5). For a front moving along the ray $y = \text{const}$, reversal means satisfaction of the equality $\partial x/\partial x_0 = 0$, a necessary condition for which is

$$\frac{\partial u}{\partial x_0} = -\frac{2}{\gamma + 1}\left(\frac{1}{t} - \frac{\delta}{t^2}(x - x_0)\right) \tag{1.7}$$

It follows from this expression that, if $\delta = 0$, reversal of the front will occur only in the case of compressive perturbations, and the earlier (i.e. for smaller t) the larger the absolute value of the velocity gradient. In the case when $\delta \neq 0$ (as can be seen from (1.7)), again, only a compressive perturbation will undergo reversal, and for perturbations propagating downstream ($x > x_0$), reversal will take place more rapidly (at the same initial amplitude of the perturbation). Analysis based on the LRT equation, however, takes no account whatever of downstream-propagating perturbations.

Thus, the use of the modified triple-deck model provides a better description of unsteady and non-linear phenomena in the flow field.

2. THE STABILITY OF A BOUNDARY LAYER INTERACTING WITH UNSTEADY INVISCID FLOW AT TRANSONIC VELOCITIES

Let us assume, using the well-known model of [9], that the perturbations of the boundary layer are periodic in the limit t and the coordinate x, in the same direction as the velocity of the unperturbed flow; the dependence on the transverse coordinate y is arbitrary:

$$f(y)\exp(\omega t + ikx) \tag{2.1}$$

The parameters of the unperturbed flow are taken as

$$u = y, \quad v = 0, \quad p = \text{const}$$

Previously, simulation of an outer unsteady inviscid flow made use of the linear LRT equation [9]. In what follows it is proposed to use the modified linear LRT equation (1.3) with the parameter K^* replaced by the parameter $K_\infty = (M_\infty^2 - 1)\delta^{-1}$,

$$\delta\frac{\partial^2\phi}{\partial t^2} + 2\frac{\partial^2\phi}{\partial t\partial x} + K_\infty\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} = 0 \tag{2.2}$$

for supersonic flow $K_\infty > 0$, for subsonic flow $K_\infty < 0$, while $K_\infty = 0$ corresponds to flow at the velocity of sound.

The equations for the lower decks are identical with those obtained previously [9]. In addition, it will be assumed, for simplicity, that the dependent and independent variables have been transformed, enabling us to eliminate the coefficient 2 of the mixed derivative in Eq. (2.2); we will, however, retain the same notation for the transformed variables as for the original ones.

Substituting perturbations of the form (2.1) into the boundary-layer equation [9], solving the resulting linearized system and using the modified equation (2.2) with appropriate boundary conditions, we obtain a modified DR

$$F(\Omega) = i^{1/3}k^{7/3}/\sqrt{\delta\omega^2 + ik(\omega + ikK_\infty)} \tag{2.3}$$

where

$$F(\Omega) = \frac{d\mathrm{Ai}}{d\Omega}\Big/\left(\int_\Omega^\infty \mathrm{Ai}(\zeta)d\zeta\right), \quad \Omega = \omega(ik)^{-2/3}$$

and $\mathrm{Ai} = \mathrm{Ai}(\Omega)$ is the Airy function.

If $\delta = 0$, equality (2.3) gives the DR (2.3) of [9].

If the outer flow is assumed to be steady (we put $\omega = 0$ on the right-hand side of (2.3)), the DR becomes

$$F(\Omega) = -(ik)^{4/3}/\sqrt{K_\infty} \tag{2.4}$$

for $K_\infty = 1$ (model supersonic flow), corresponding to the results of [4]. If $K_\infty = -1$ (model subsonic flow)

$$F(\Omega) = -i^{1/3}k^{4/3} \tag{2.5}$$

The DR obtained in [5] for subsonic outer flow corresponds to Eq. (2.5) with $k > 0$. Analysis of Eq. (2.5) is sufficient to determine the roots of the DR for subsonic outer flow (see Terent'yev's dissertation, cited earlier as a footnote). Thus, Eq. (2.3) contains (as special cases) the DRs for the problems of stability of the boundary layer in free viscid–inviscid interaction under the various conditions (steady and unsteady, at subsonic, transonic and supersonic velocities) that have been considered previously [4–9]. It should be noted here that formula (2.4) is not suitable for analysing the stability of the boundary layer in the case of a flow at the velocity of sound $K_\infty = 0$ (this was noted in Terent'yev's dissertation), whereas formula (2.5) and its special case with $\delta = 0$ do not have this limitation.

The expressions on the left of relations (2.3)–(2.5) may be expanded in infinite series in powers of ω. Confining ourselves to a finite number of terms of the series, we obtain an algebraic equation of the relevant degree. The roots of such equations correspond to waves propagating in the boundary layer.

Let us consider the asymptotic behaviour of the roots of Eq. (2.3). As $\Omega \to \infty$ (for example, the high-frequency approximation $\omega \to \infty$, $k \to \infty$), the expression on the right of Eq. (2.3) may be expanded as follows:

$$F(\Omega) = -\Omega - 1/\sqrt{\Omega} + O(1/\Omega^2) \tag{2.6}$$

In that case, it follows from (2.3) that

$$\omega + ik/\sqrt{\omega} = -ik^3/\sqrt{\delta\omega^2 + ik(\omega + ikK_\infty)} \tag{2.7}$$

and from (2.5) that

$$\omega + ik/\sqrt{\omega} = -ik^2 \tag{2.8}$$

Let us ignore the term $ik/\sqrt{\omega}$ on the left of Eqs (2.7) and (2.8). It then follows at once from Eq. (2.8) that in the principal terms $\omega = -ik^2$. Organising an iterative process, we find that

$$\omega = -ik^2 + i\sqrt{i}$$

so that

$$\omega_\pm = -ik^2 \mp \sqrt{2}/2 \pm i\sqrt{2}/2$$

The root ω_- was determined previously [5, 9]; it corresponds to a perturbation which increases with time.

Equation (2.7), in turn, may be transformed to a fourth-order equation in ω

$$\delta\omega^4 + ik\omega^3 - k^2K_\infty\omega^2 + k^6 = 0 \tag{2.9}$$

If $\delta = 0$, this becomes a third-order equation, and it is therefore obvious that one of the roots of Eq. (2.9) disappears when $\delta = 0$. Thus, the modified model describes one perturbation wave more than the Ryzhov–Savenkov model [9]. The latter, in turn, describes one perturbation wave more than the model that presupposes a steady outer flow [7].

Let k be a real number and suppose $\omega = -i\bar{\omega}$. Then Eq. (2.9) may be transformed to an equation with real coefficients

$$\bar{\omega}^4 - \frac{k}{\delta}\bar{\omega}^3 + \frac{k^2K_\infty}{\delta}\bar{\omega}^2 + \frac{k^6}{\delta} = 0 \tag{2.10}$$

To find its roots for $\delta = 0$, we introduce the notation

$$\chi_1 = \frac{k^2}{2} + \frac{K_\infty^3}{27}, \quad \chi_2 = \chi_1^2 - \frac{K_\infty^6}{8}, \quad \chi^\pm = (\chi_1 + \chi_2^{1/2})^{1/3} \pm (\chi_1 - \chi_2^{1/2})^{1/3}$$

If $\chi_2 > 0$, we have one real root and two complex-conjugate roots

$$\bar{\omega}_1 = k\chi^+ + \frac{kK_\infty}{3}, \quad \bar{\omega}_{2,3} = -\frac{k}{2}\chi^+ + \frac{kK_\infty}{3} \pm \frac{i\sqrt{3}}{2}k\chi^-$$

For the leading terms of degree k, the real root of the equation is

$$\bar{\omega}_1 = k^{5/3} + \frac{1}{3}kK_\infty \tag{2.11}$$

which, in view of the substitution $\omega = -i\bar{\omega}$, is identical with the first two terms of formula (3.3) in [9].

Clearly, the root (2.11) is of order $k^{5/3}$. The principal term $\omega = -ik^{5/3}$ for ω may be substituted into the previously omitted term $ik/\sqrt{\omega}$, and the order of that term determined (by Newton's method). We have

$$ik/\sqrt{\omega} = O(k^{1/6})$$

Thus, the order of the omitted term is less than that of the remaining ones, which justifies ignoring the former, as was down previously deriving DR (2.9).

For the complex roots we have

$$\bar{\omega}_{2,3} = -k^{5/3}/2 + kK_\infty/3 \pm i\sqrt{3}k^{5/3}/2 \tag{2.12}$$

It is obvious that the increment of the perturbations equals $\sqrt{2}k^{1/6}/3$ (unlike the result of [9]); but, as before, the perturbations are amplified in both subsonic and supersonic outer flow (i.e. irrespective of the sign of K_∞). As remarked previously, earlier studies [4, 5] demonstrated the growth of perturbations in subsonic flow only.

If $\chi_2 = 0$, Eq. (2.10) has three real roots

$$\bar{\omega}_1 = 2k\chi^{1/3} + kK_\infty/3, \quad \bar{\omega}_2 = \bar{\omega}_3 = -k\chi^{1/3} + kK_\infty/3 \tag{2.13}$$

If $\chi_2 < 0$, one has the irreducible case (also with three real roots), and the following formulae may be used for the roots of Eq. (2.10)

$$\bar{\omega}_1 = \frac{2}{3}kK_\infty\cos\frac{\alpha}{3}, \quad \bar{\omega}_2 = \bar{\omega}_3 = -\frac{2}{3}kK_\infty\cos\left(\frac{\alpha}{3} \pm \frac{\pi}{3}\right); \quad \cos\alpha = 1 + \frac{27k^3}{2K_\infty^3} \tag{2.14}$$

Since the perturbations are assumed to have the form (2.1), we have

$$\exp(\omega t) = \exp(-i\bar{\omega}t)$$

and therefore the presence of a root $\bar{\omega}$ with positive imaginary part implies exponential growth of perturbations and instability of the flow. Exactly the same situation is observed in the case $\chi_2 > 0$: one of the roots leads to instability. The condition $\chi_2 > 0$ is satisfied if

$$k^2 > K_\infty^3(1/\sqrt{2} - 2/27) \tag{2.15}$$

If $K_\infty > 0$ (for supersonic flow), condition (2.15) is satisfied for sufficiently larger k; if $K_\infty \le 0$, it is satisfied for any k (that is, it is always satisfied for flows at subsonic velocities and the velocity of sound). The latter case agrees with the previously used high-frequency approximation ($\omega, k \to \infty$).

In the case of the real roots (2.13), (2.14), the flow is stable. That situation occurs in supersonic flow when the sign of inequality (2.15) is reversed, in which case the admissible value of k increases as the Mach number increases (in supersonic flow corresponding to the increase in K_∞).

A simple check shows that the roots of Eq. (2.10) are not roots of the equation obtain from (2.10) with $\delta = 0$. Thus, the modified model yields another value even for all other perturbations (corresponding to those defined previously [9]) in the flow field.

If $\delta \ne 0$, Eq. (2.10) may be solved by Ferrari's method [16]. Rather cumbersome calculations yield four roots y_1, y_2, y_3 and y_4, such that for $\delta \to 0$, $y_1 = k/\delta + O(1)$, $y_2, y_3, y_4 = O(1)$. If one takes $\delta = 0$, the first root disappears, as in the case of unsteady transonic aerodynamics, where weak perturbations are simulated using the LRT equation (see above, Section 1).

Let us consider the special case in which Eq. (2.10) is represented in the form

$$\left(\bar{\omega} - \frac{k}{\delta}\right)(\bar{\omega}^3 - \delta k^4\bar{\omega} - k^5) + \frac{k^2\bar{\omega}^2}{\delta}(K_\infty + \delta^2k^2) = 0$$

Obviously, if $K_\infty = -\delta^2k^2$, it has the root

$$\bar{\omega}_1 = k/\delta$$

corresponding to a wave propagating downstream ($\omega/k > 0$, $\omega < 0$) without change of amplitude. This case corresponds to viscid–inviscid interaction at subsonic velocities and does not contradict the proposed asymptotic representation (when $k = O(1/\delta)$, $|K_\infty| = O(1)$). The other roots when $K_\infty = -\delta^2k^2$ are determined form the equation

$$\bar{\omega}^3 - \delta k^4\bar{\omega} - k^5 = 0 \tag{2.16}$$

which (as follows, e.g. from Vieta's formula) does not have roots that increase without limit as $\delta \to 0$.

Equation (2.16) differs from Eq. (2.10) when $\delta = 0$ in having no quadratic term and having a linear term. We introduce the following notation

$$\sigma = \frac{k^{10}}{4} - \frac{\delta^3k^{12}}{27}, \quad \sigma_\pm = \left(\frac{k^5}{2} \pm \sqrt{\sigma}\right)^{1/3}$$

and write the roots of Eq. (2.16) for $\sigma > 0$ as

$$\bar{\omega}_2 = \sigma_+ + \sigma_-, \quad \bar{\omega}_{3,4} = -(\sigma_+ + \sigma_-)/2 \pm i\sqrt{3}(\sigma_+ - \sigma_-)/2$$

As $\delta \to 0$, $1/\delta = O(k)$, these roots become identical with the roots (2.11), (2.12) in the principal approximation with respect to k.

The singular root ($\omega = O(1/\delta)$) may be determined approximately from Eq. (2.7) by substituting $\omega = -ik/\delta$ for $\sqrt{\omega}$. We have

$$\omega = -ik/\delta + (1-i)\sqrt{k\delta/2}$$

The perturbation corresponding to this root increases with time; if $1/\delta = O(k)$, this increase corresponds exactly to the case of interaction at subsonic velocities.

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